

Markovian semigroup from non-Markovian evolutions

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It is shown that a convex combination of two non-Markovian evolutions may lead to Markovian semigroup. This shows that convex combination of quantum evolutions displaying nontrivial memory effects may result in a perfectly memoryless evolution.

PACS numbers: 03.65.Yz, 03.65.Ta, 42.50.Lc

Introduction – A general quantum evolution is represented by a dynamical map $\Lambda(t)$, i.e. a family of completely positive and trace-preserving maps such that $\rho \rightarrow \rho(t) = \Lambda(t)\rho$, where $\rho(t)$ denotes the density operator at time ‘ t ’ [1]. One usually assumes that $\Lambda(t)$ satisfies time-local master equation

$$\frac{d}{dt}\Lambda(t) = \mathcal{L}(t)\Lambda(t), \quad (1)$$

where the time-local generator $\mathcal{L}(t)$ has the following well-known form

$$\begin{aligned} \mathcal{L}(t)\rho = & -i[H(t), \rho] \\ & + \sum_{\alpha} \gamma_{\alpha}(t) \left(V_{\alpha}(t)\rho V_{\alpha}^{\dagger}(t) - \frac{1}{2}\{V_{\alpha}^{\dagger}(t)V_{\alpha}(t), \rho\} \right), \end{aligned} \quad (2)$$

with the time dependent Hamiltonian $H(t)$ and time dependent dissipative part governed by time dependent rates $\gamma_{\alpha}(t)$ and noise operators $V_{\alpha}(t)$. Recently a lot of attention was devoted to the analysis of Markovianity of quantum evolution represented by $\Lambda(t)$ (see recent review papers [2, 3]). Let us recall that $\Lambda(t)$ is called divisible (or CP-divisible) if

$$\Lambda(t) = V(t, s)\Lambda(s), \quad (3)$$

and $V(t, s)$ is completely positive for all $t \geq s$ [4, 5]. This property is fully characterized by the time-local generator: $\Lambda(t)$ is CP-divisible if and only if $\gamma_{\alpha}(t) \geq 0$ for all $t \geq 0$. One of the approaches to quantum Markovianity states that quantum evolution is Markovian iff the corresponding dynamical map $\Lambda(t)$ is CP-divisible [4, 5]. A slightly weaker notion of Markovianity was proposed in [6]. The advantage of BLP approach [6] is an operational characterization based on the following definition: $\Lambda(t)$ is Markovian if for any ρ_1 and ρ_2

$$\frac{d}{dt} \|\Lambda(t)[\rho_1 - \rho_2]\|_1 \leq 0, \quad (4)$$

where $\|X\|_1 = \text{Tr}\sqrt{X^{\dagger}X}$ denotes the trace norm of X . In this paper we attribute Markovianity to the notion of

CP-divisibility. However, the main example we use to illustrate the paper does not distinguish between these two notions.

Convex combination of Markovian evolutions – Note that if $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ are Markovian generators then $\alpha_1\mathcal{L}_1(t) + \alpha_2\mathcal{L}_2(t)$ is again Markovian generator for arbitrary $\alpha_1, \alpha_2 \geq 0$. Hence Markovian generators define a convex set (actually a convex cone) in the space of all admissible time-local generators (2). It is no longer true on the level of dynamical maps, i.e. if $\Lambda_1(t)$ and $\Lambda_2(t)$ are Markovian (i.e. CP-divisible), then $\alpha_1\Lambda_1(t) + \alpha_2\Lambda_2(t)$ needs not be CP-divisible [4]. A simple example illustrating that the space of CP-divisible maps is not convex was recently provided in [7]: consider two Markovian semigroups generated by

$$\mathcal{L}_1\rho = \frac{c}{2}[\sigma_1\rho\sigma_1 - \rho] \quad ; \quad \mathcal{L}_2\rho = \frac{c}{2}[\sigma_2\rho\sigma_2 - \rho], \quad (5)$$

where $c > 0$, and σ_1, σ_2 are Pauli matrices. One finds for the convex combinations $\Lambda(t) = \frac{1}{2}[e^{t\mathcal{L}_1} + e^{t\mathcal{L}_2}]$

$$\Lambda(t)\rho = \frac{1 + e^{-ct}}{2}\rho + \frac{1 - e^{-ct}}{4}(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2). \quad (6)$$

Clearly $\Lambda(t)$ is a legitimate dynamical map but it is not CP-divisible. Indeed, the corresponding time-local generator reads

$$\mathcal{L}(t)\rho = \sum_{k=1}^3 \gamma_k(t)[\sigma_k\rho\sigma_k - \rho], \quad (7)$$

where

$$\gamma_1(t) = \gamma_2(t) = \frac{c}{2}, \quad \gamma_3(t) = -\frac{c}{2}\tanh(ct), \quad (8)$$

and evidently leads to non-Markovian evolution due to $\gamma_3(t) < 0$. This generator was analyzed in [8] as an example of *eternal* non-Markovianity. This simple example shows that convex combination of Markovian evolutions (even semigroups!) might lead to legitimate non-Markovian evolution. Interestingly, this example may be easily generalized for qudit systems [7].

Convex combination of non-Markovian evolutions – In the present paper we provide a simple example showing that a convex combination of two non-Markovian evolution may lead to Markovian semigroup. Consider a quantum channel \mathcal{E} for a qudit system which defines a projector, that is, $\mathcal{E}^2 = \mathcal{E}$. A typical example is a channel which maps arbitrary state ρ into a fixed state ω , that is, $\mathcal{E}\rho = \omega \text{Tr}\rho$. If $\omega = \frac{1}{d}\mathbb{I}$ then \mathcal{E} is a completely depolarizing channel. Taking an orthonormal basis $\{|1\rangle \dots, |d\rangle\}$ in \mathbb{C}^d one may define another CPTP projector via $\mathcal{E}\rho = \sum_{k=1}^d |k\rangle\langle k|\rho|k\rangle\langle k|$. Now, for arbitrary CPTP projector \mathcal{E} let us consider the following Markovian generator

$$\mathcal{L} = \mathcal{E} - \mathbb{I}. \quad (9)$$

We show that for a given $\gamma > 0$ one can find time dependent $\gamma_1(t)$ and $\gamma_2(t)$ such that the following Markovian semigroup

$$\Lambda(t) = e^{\gamma\mathcal{L}t} = e^{-\gamma t}\mathbb{I} + [1 - e^{-\gamma t}]\mathcal{E},$$

may be constructed as a convex combination

$$\Lambda(t) = p\Lambda_1(t) + (1-p)\Lambda_2(t), \quad (10)$$

with

$$\Lambda_k(t) = \exp(\Gamma_k(t)\mathcal{L}) = e^{-\Gamma_k(t)}\mathbb{I} + [1 - e^{-\Gamma_k(t)}]\mathcal{E}, \quad (11)$$

and $\Gamma_k(t) = \int_0^t \gamma_k(\tau)d\tau$. Moreover, neither $\Lambda_1(t)$ nor $\Lambda_2(t)$ is Markovian which means that $\gamma_k(t) \not\geq 0$.

Note, that $\Lambda_k(t)$ is CP if and only if $\mu_k(t) = e^{-\Gamma_k(t)} \in [0, 1]$. One has the following relation

$$e^{-\gamma t} = p\mu_1(t) + (1-p)\mu_2(t). \quad (12)$$

Let $\mu_1(t) = \frac{1}{p}e^{-\gamma t}g(t)$ and hence

$$\mu_2(t) = e^{-\gamma t} \frac{1 - g(t)}{1 - p}.$$

The initial condition $\Lambda_k(0) = \text{id}$ implies $\mu_1(0) = \mu_2(0) = 1$, hence $g(0) = p$. Moreover, the constraint $0 < \mu_1(t) \leq 1$ implies $0 < g(t) \leq p$. Now, to satisfy $0 < \mu_2(t) \leq 1$ let us consider the following $g(t)$: $g(t) = p$ for $t \in [0, t_*]$ and $p(t) \in [0, p]$ for $t > t_*$, where t_* is defined via the relation

$$e^{-\gamma t_*} = 1 - p,$$

which implies $t_* = -\frac{1}{\gamma} \ln(1 - p)$. Finally,

$$\mu_1(t) = \begin{cases} e^{-\gamma t} & t \in [0, t_*] \\ e^{-\gamma t} \frac{g(t)}{p} & t > t_* \end{cases} \quad (13)$$

and

$$\mu_2(t) = \begin{cases} e^{-\gamma t} & t \in [0, t_*] \\ e^{-\gamma t} \frac{1 - g(t)}{1 - p} & t > t_* \end{cases} \quad (14)$$

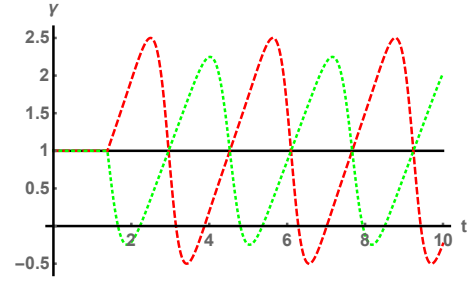


FIG. 1: Local decoherence rates for three types of dynamical maps with $\gamma = 1$, $\varepsilon = \frac{3}{4}$ and $p = \frac{3}{4}$. The black line represents Markovian semigroup. The dashed (red) and the dotted (green) are $\gamma_1(t)$ and $\gamma_2(t)$ respectively, and due to its interval negativity, they represent non-Markovian dynamics.

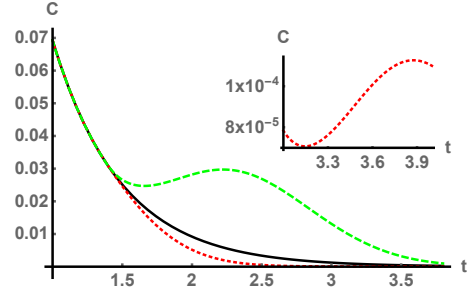


FIG. 2: Channel capacity C of three dynamical maps with $\gamma = 1$, $\varepsilon = \frac{3}{4}$ and $p = \frac{3}{4}$. The black line corresponds to Markovian semigroup while the dashed (red) and the dotted (green) are channel capacities for non-Markovian dynamics $\Lambda_1(t)$ and $\Lambda_2(t)$ respectively. In the green, we may clearly observe nonmonotonic behaviour. In the upper right corner we may observe nonmonotonicity of $\Lambda_2(t)$ for $t \in (3, 4)$. Therefore, both of $\Lambda_1(t)$ and $\Lambda_2(t)$ are non-Markovian.

The corresponding local depolarizing rates read

$$\gamma_1(t) = \gamma - \frac{\dot{g}(t)}{g(t)}, \quad \gamma_2(t) = \gamma + \frac{\dot{g}(t)}{1 - g(t)}$$

and hence for $t \leq t_*$ one has $\gamma_1(t) = \gamma_2(t) = \gamma$. As an example let us consider

$$g(t) = p \{1 - \varepsilon \sin^2(\gamma[t - t_*])H(t - t_*)\}, \quad (15)$$

where $H(t)$ denotes the Heaviside step function, and $0 < \varepsilon < 1$. The corresponding local depolarizing rates for $\gamma = 1$, $\varepsilon = \frac{3}{4}$ and $p = \frac{3}{4}$ are displayed in Fig. 1. It is evident that both $\gamma_1(t)$ and $\gamma_2(t)$ become negative for some time intervals and hence neither $\Lambda_1(t)$ nor $\Lambda_2(t)$ is CP-divisible. It is worth stressing that manipulation of parameters γ, ε and p can give us different types of behaviour, from CP-divisible for both $\Lambda_1(t)$ and $\Lambda_2(t)$ to non-divisible for either or both of $\Lambda_1(t)$ and $\Lambda_2(t)$.

Non-Markovianity of quantum evolution represented by $\Lambda(t)$ may be also analyzed in terms of channel capacity [9]. If $\Lambda(t)$ is CP-divisible then

$$\frac{d}{dt}C(\Lambda(t)) \leq 0, \quad (16)$$

i.e. capacity monotonically decreases. For depolarising channels one may easily evaluate channel capacity [10]:

for $\mathcal{E}_\lambda = \lambda \mathbb{1} + (1 - \lambda)\mathcal{E}$ one has

$$\mathcal{C}(\mathcal{E}_\lambda) = \ln d - S_{\min}(\mathcal{E}_\lambda), \quad (17)$$

where the minimal output entropy reads

$$\begin{aligned} S_{\min}(\mathcal{E}_\lambda) &= - \left(\lambda + \frac{1 - \lambda}{d} \right) \ln \left(\lambda + \frac{1 - \lambda}{d} \right) \\ &\quad - (d - 1) \frac{1 - \lambda}{d} \ln \frac{1 - \lambda}{d}. \end{aligned}$$

The corresponding plots of capacities in the qubit case are provided in Fig. 2. It is evident that $\Lambda_k(t)$, $k = 1, 2$ displays highly non-Markovian behaviour.

Semi-Markov evolution – Quantum evolution generated by the time-local generator $\mathcal{L}(t) = \gamma(t)[\mathcal{E} - \mathbb{1}]$, with \mathcal{E} being a CPTP projector may be equivalently described in terms of non-local memory kernel

$$\mathcal{K}(t) = k(t)[\mathcal{E} - \mathbb{1}], \quad (18)$$

for some memory function $k(t)$ [11, 12]. The corresponding non-local master equation

$$\frac{d}{dt}\Lambda(t) = \int_0^t \mathcal{K}(t - \tau)\Lambda(\tau)d\tau, \quad (19)$$

gives rise to the following solution

$$\Lambda(t) = \left(1 - \int_0^t f(\tau)d\tau \right) \mathbb{1} + \int_0^t f(\tau)d\tau \mathcal{E}, \quad (20)$$

and the function $f(t)$ is related to the memory function $k(t)$ via

$$\tilde{k}(s) = \frac{s\tilde{f}(s)}{1 - \tilde{f}(s)}, \quad (21)$$

where $\tilde{f}(s) = \int_0^\infty e^{-st}f(t)dt$ denotes the corresponding Laplace transform. One calls $\Lambda(t)$ *semi-Markov* if $f(t) \geq 0$ and $\int_0^\infty f(\tau)d\tau \leq 1$. In this case $f(t)$ plays the role of so-called waiting time distribution and $1 - \int_0^\infty f(\tau)d\tau$ is interpreted as so-called survival probability. Interestingly, it is well known that in the class (20) the evolution $\Lambda(t)$ is semi-Markovian if and only if it is Markovian (CP-divisible) [11]. Moreover, $\Lambda(t)$ defined by (20) defines a semigroup iff $f(t) = \gamma e^{-\gamma t}$. Our example shows that convex combination of two evolutions which are not semi-Markov, i.e. $f_1(t), f_2(t) \not\geq 0$, and hence non-Markovian, may results in Markovian semigroup:

$$pf_1(t) + (1 - p)f_2(t) = \gamma e^{-\gamma t}, \quad (22)$$

for $t \geq 0$. Note, that (22) reproduces (12), that is, $f_k(t) = \gamma\mu_k(t)$.

Conclusions — A set of Markovian (CP-divisible) evolutions is not convex. It is shown that a convex combination of two non-Markovian evolutions may lead to Markovian semigroup. Similarly, using memory kernel master equation we shown that convex combination of quantum evolutions which are not semi-Markov (and hence non-Markovian) may result in Markovian semigroup. This shows that convex combination of quantum evolutions displaying nontrivial memory effects may kill all memory effects and result in a perfectly memoryless evolution.

Acknowledgements — D.C. was partially supported by the National Science Center project 2015/17/B/ST2/02026.

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- [1] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford Univ. Press, Oxford, 2007).
 - [2] Á. Rivas, S.F. Huelga, and M.B. Plenio, Rep. Prog. Phys. **77**, 094001 (2014).
 - [3] H.-P. Breuer, E.-M. Laine, J. Piilo, B. Vacchini, *Non-Markovian dynamics in open quantum systems*, arXiv: 1505.01385 (2015).
 - [4] M. M. Wolf, J. Eisert, T. S. Cubitt and J. I. Cirac, Phys. Rev. Lett. **101**, 150402 (2008).
 - [5] Á. Rivas, S.F. Huelga, and M.B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).
 - [6] H.-P. Breuer, E.-M. Laine, J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
 - [7] D. Chruściński, and F. A. Wudarski, Phys. Rev. A **91**, 012104 (2015).
 - [8] E. Andersson, J. D. Cresser, and M. J. W. Hall, Phys. Rev. A **89**, 042120 (2014).
 - [9] B. Bylicka, D. Chruściński, and S. Maniscalco, Scientific Reports, **4**, 5720 (2014).
 - [10] C. King, IEEE Trans. Inf. Theory, **49**, 221 (2003).
 - [11] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. **13**, 093004 (2011).
 - [12] D. Chruściński and A. Kossakowski, EPL **97**, 20005 (2012).